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Rapid Note

Damage-spreading in the parallel Bak-Sneppen model

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Abstract. We study the behavior under perturbations of the Parallel Bak-Sneppen model (PBS) in $1+1$ dimension, which has been shown to belong to the universality class of Directed Percolation (DP) in $1+1$ dimensions [1]. We focus our attention on the damage-spreading features of the PBS model with both random and deterministic updating, which are studied and compared to the known results for the extremal Bak-Sneppen model (BS) and for DP. For both random and deterministic updating, we observe a power law growth of the Hamming distance. In addition, we compute analytically the asymptotic plateau reached by the distance after the growing phase.

PACS. 05.20.-y Statistical mechanics – 05.45.+b Theory and models of chaotic systems – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes

A great deal of evidence has been put forward in recent years for the appearance of power law statistics in nature: a wide variety of phenomena, from earthquakes [2,3] to biological evolution [4], from surface growth [5] to fluid displacement in porous media [6], exhibit scale invariance in both space and time. To explain these all-pervading power-law tails, Bak, Tang, and Wiesenfeld introduced the concept of self-organized criticality (SOC) [7]. In a nutshell, SOC means that certain driven spatially extended systems evolve spontaneously towards a critical globally stationary dynamical state with no characteristic time or length scales [8]. This scale invariance implies that the correlation length in these systems is infinite and consequently a small (local) perturbation can produce a global (maybe even drastic) effect. This possibility leads naturally to the study of the sensitivity to perturbations in (self-organized) critical systems.

To study the propagation of local perturbations (*damage spreading*) one can borrow a technique from dynamical systems theory. Let us consider, for instance, two copies of the same dynamical system, with slightly different initial conditions. By following the dynamics of both copies and studying the evolution in time of the "distance" $d(t)$ between them, it is possible to quantify the effect of the initial perturbation. Indeed, assuming that the distance $d(t)$ grows exponentially, and defining the Lyapunov exponent λ *via*

$$d(t) = d_0 \exp(\lambda\, t), \tag{1}$$

three different behaviors can be distinguished, corresponding to λ being either positive, negative or zero. The case $\lambda > 0$ corresponds to the so-called *chaotic* systems, where the extremely high sensibility to initial conditions leads to exponentially diverging trajectories developing on a chaotic or *strange* attractor. The case $\lambda < 0$, instead, characterizes those systems in which the dynamics has a simple attractor such as a fixed point or a limit cycle and any initial perturbation is "washed out" with exponential rapidity.

The boundary case, $\lambda = 0$, admits, in turn, a whole class of functions $d(t)$, namely

$$d(t) \sim t^{\alpha}\,. \tag{2}$$

where α is some exponent, characteristic of the system. In particular, $\alpha > 0$ corresponds to weak sensitivity to initial conditions while $\alpha < 0$ corresponds to weak insensitivity to initial conditions (as an example, the reader is referred to Ref. [9], where this analysis is performed for the logistic map at its critical point [10]).

When performed on the BS model [4], this analysis shows that the critical properties of the model allow us to describe its behavior under perturbations *via* equation (2), with an exponent $\alpha = 0.32$ [12,13]. Recently [15], the results presented in [12] were explained by relating the BS model to a simpler toy model. In this paper we analyze the behavior under perturbations of the PBS model, introduced not long ago by Sornette and Dornic [1]. We study two versions of it, namely the one with random updating and the one with logistic updating, and discuss of the differences and similarities between the BS and the PBS models, from the point of view of damage spreading.

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Originally proposed to describe ecological evolution, the BS model describes an ecosystem as a collection of N species on a one dimensional lattice. To each species corresponds a fitness described by a number f between 0 and 1. The initial state of the system is defined by assigning to each site j a random fitness f_j^0 chosen from a uniform distribution. The dynamics proceeds sequentially by mutating at each time-step the less fit species together with its two nearest neighbours. In the parallel version of the BS model, the dynamics proceeds with a parallel updating, according to the following rules [1,16].

1. Find the site with the absolute minimum fitness f_{min} on the lattice (the active site) and its two nearest neighbours.
2. Update the values of their fitness by assigning to them new random numbers from a uniform distribution.
3. Search for all sites on the lattice with fitness $f < f_{min}$ and update them together with their nearest neighbours. Repeat the search until there are no sites left with $f < f_{min}$.
4. Return to step 1.

The difference between the PBS model and the original BS model is in step 3. In the extremal version, once the minimum and its nearest neighbours are updated one looks for the new minimum, and consequently the number of updated sites per time step, U_t, is constant ($U_t \equiv 3$). In the parallel version, instead, this number will follow a complex temporal evolution during an avalanche, and in general ($U_t \geq 3$). In fact, the distribution of the number of updated sites per time step inside an avalanche shows a nearly flat distribution, with a upper cutoff at a value which is comparable with the system size [14]. Due to this saturation, one observes that the distance $D(t)$ (see below for a definition) grows in time much faster than in the BS case and that finite size effects are also much stronger. Furthermore, the PBS can be exactly mapped onto the Directed Percolation (DP) problem [1], where the avalanche time distribution in the PBS model is equivalent to the cluster distribution in DP and the threshold for PBS is equivalent to the critical probability in DP.

In both versions of the BS model, after an initial transient that will be of no interest to us here, a non-trivial critical state is reached. This critical state, characterized by its statistical properties, can be understood as the *fluctuating balance* between two competing "forces". Indeed, while the random assignation of the values, together with the coupling, acts as an entropic disorder, the mutation of all those $f < f_{min}$ acts as an ordering force. As a result of this competition, at the stationary state the majority of the f_j have values above a certain threshold f_c. Only a few will be below f_c, namely those belonging to the running avalanche (see [4,17,18,21] for a detailed discussion). The value of f_c represents the "equilibrium" between order and disorder. To change this critical value one should consider a mechanism that acts in one of the two competing factors. The introduction of time correlations in the values of consecutive f_j for instance, produces an increasing in f_c, while an increase in the number of nearest neighbours decreases f_c [18]. In the original version of the BS model, with random uncorrelated updating and three nearest neighbours, $f_c \approx 0.6607$. Since the disorder is stronger in the parallel version ($U_t \geq 3$), one expects the equilibrium point to be displaced towards the completely disordered value. In fact, for the parallel BS model, $f_c \approx 0.5371(1)$ [1]. This results can be obtained both numerically and analytically by mapping the model onto directed percolation.

Table 1. Values of the exponent α for different sizes and updating rules.

N	100	250	500	750	2000
α ran. up.	0.44(1)	0.47(1)	0.48(1)	0.48(1)	0.47(1)
α log. up.	0.43(1)	0.46(1)	0.47(1)	0.47(1)	0.47(1)

As mentioned above, to study the behavior under perturbations we produce two identical copies B_1 and B_2 of the system of size N in the critical state, and find the minimum (the active site). Then, we introduce a slight perturbation in B_2 (as explained later on) and follow the evolution in time of the Hamming distance (to distinguish between extended systems and maps we will hereafter use the capital D for the distance)

$$D(t) = \frac{1}{N}\sum_{j=1}^{N} |f_j^1 - f_j^2| . \quad (3)$$

Since this quantity has strong fluctuations, we consider the average $\langle D(t) \rangle$ over different realizations of the initial values of the f_j. In particular all the simulations presented here are the result of averaging over 5×10^2 realizations. The simulations for the Hamming distance (3) are performed with both random and deterministic (logistic) updating rules, with system sizes $N = 100, 250, 500, 750, 2000$. Let us begin by discussing the results obtained for random updating. As explained in [15], $D(t)$ may depend on the internal correlation of the system and on the correlations between the two copies. In the 1D BS model, due to the choice of unit of time, which allows only a number O(1) of sites to be updated, the growth rate must give an exponent $\alpha < 1$ and stop at a certain time $\tau \sim N^z$, at which a crossover to a saturation regime appears. Clearly, this is due to the fact that after τ time-steps each site of the lattice has been covered at least once. For $t \gg \tau$, almost all the lattice sites have been covered and the two strings are made of the same random numbers placed in different position along the lattice (see [15] for a detailed discussion).

In the PBS case, $D(t)$ reaches, after an initial power law growth (as in the BS case), a well-defined plateau (see Fig. 1). However, due to the faster parallel dynamics, the value of the exponent α is expected to be larger than the one for the the extremal non-parallel case. The values α measured for different system sizes are shown in Table 1. As expected, these values are larger than in the original BS model. Moreover, there seem not to be dependence of the exponent α on the system size N. Notice that our exponent differs from the one obtained in [20] for the Domany-Kinzel model in the context of DP, onto which the PBS

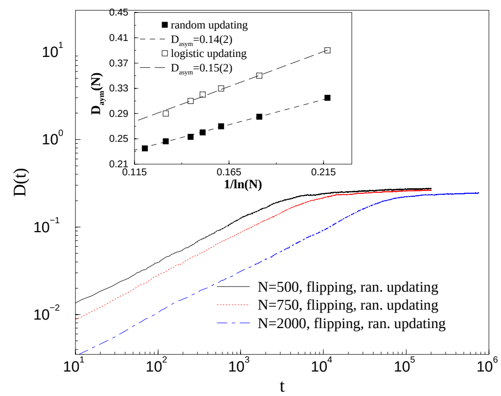

Fig. 1. $\log_{10}$-$\log_{10}$ plot of the distance $D(t)$ in the PBS model, *versus* the time t, for random updating. Inset: logarithmic extrapolation of the plateau as a function of N, for both random and logistic updating rules. The infinite size value D_{asym} obtained is in agreement with our analytical estimate.

can be mapped. This discrepancy is due to the choice of timescale for the measure of the distance. Indeed, in the Domany-Kinzel model case only one active (occupied) site per time step can be updated, together with its neighbour, thus resulting in a dilated time scale with respect to the study presented here. Thus, in order to compare the two models one has to establish the relationship between the two timescales. This is not easy to realize for the Hamming distance. In fact, according to this interpretation, equal times for the two copies on the Monte-Carlo parallel timescale are not equal times on the DP-like timescale. To circumvent this problem, we realized a set of PBS simulations (with system size $N = 2000$) for a single copy, computing at every Monte-Carlo (parallel) time step $\delta t = 1$ the number $n_{act}(t)$ of sites below threshold (which is itself time-dependent). This defines the temporal increment for the DP-like time scale $\delta t_{DP} = \delta t \; n_{act}(t) = n_{act}(t)$. The effective DP time at the MC step t is thus connected to the effective time at MC step $t-1$ by the relation:

$$t_{DP}(t) = t_{DP}(t-1) + n_{act}(t). \qquad (4)$$

Then, we mediated over different realizations of the dynamics, obtaining the scaling of the effective DP time with the MC time of the simulation. The result, shown in Figure 2, is that $t_{DP} \sim t^{\zeta}$ with $\zeta = 1.41(2)$. From this, if we assume that t_{DP} is the equivalent of the DP timescale for PBS, we can combine the scaling law for $D(t)$ and that for t_{DP} to get the effective scaling exponent α^* for the Hamming distance with respect to the DP time-scale t_{DP}:

$$\alpha^* = \frac{\alpha}{\zeta} = \frac{0.47}{1.41} = 0.33(1). \qquad (5)$$

This value is quite near to the DP exponent $\alpha_{DP} = 0.314(1)$ [20].

At any given time step t during an avalanche, the average growth of $D(t)$ is connected to the mean number of sites $\sigma(t)$ covered by the activity in each system [15]. In Figure 3, $\sigma(t)$ exhibits a power law behavior, $\sigma(t) \sim t^{\mu}$.

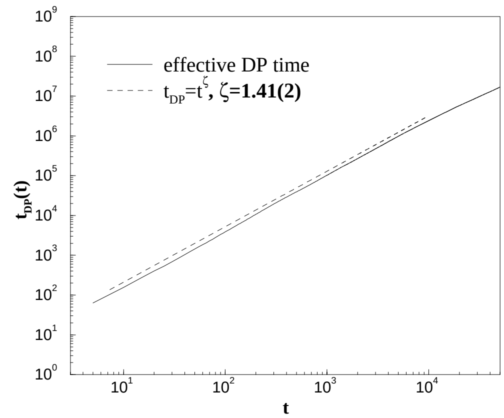

Fig. 2. Scaling of the effective DP-like time t_{DP} with the Monte-Carlo time t, on a $\log_{10}$-$\log_{10}$ scale. We get a power law behavior with exponent $\zeta = 1.41(2)$.

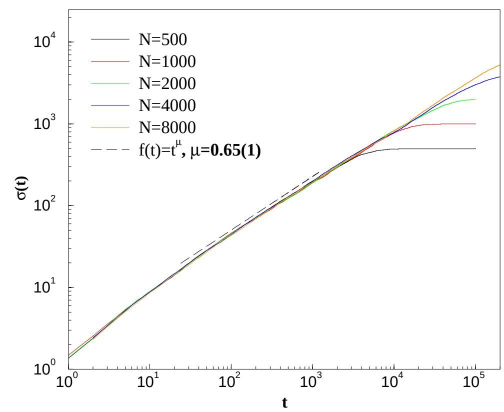

Fig. 3. $\log_{10}$-$\log_{10}$ plot of the number $\sigma(t)$ of sites covered during the evolution of a single copy of the system, *versus* the time t, for random updating rule.

The values of the scaling exponent μ at different sizes N (see Fig. 3) are always bigger than the corresponding values of the exponents for the Hamming distance. This is expected since the correlations between the two copies in $D(t)$, if present, can only decrease the value of α with respect to μ [15]. In the parallel model there are strong correlations between the two copies, due to the high number of updated sites at each time step.

The asymptotic value of the Hamming distance shows quite a strong and persistent dependence on the system size N and converges to an asymptotic value only logarithmically in N (see inset of Fig. 1). Then, in order to get the real value of the plateau, one has to go to the thermodynamic limit $N = \infty$, once the plateau has been reached. This is realized by a logarithmic extrapolation of the data for the different sizes. The value of this plateau can be obtained in terms of the asymptotic stationary fitness distribution $\rho(f)$ namely [15,19]

$$D_{asym}(N) = \langle D(t \to \infty, N) \rangle = \int_0^1 df^1 df^2 \rho_1(f^1) \rho_2(f^2) |f^1 - f^2|, \qquad (6)$$

where $\rho_i(f)$ is the normalized distribution function (at $t=\infty$) for the variables $f^i \in B_i$ [22], at a given system size N. The dependence of the plateau $D_{asym}(N)$ on N must then be related to the shape of the stationary fitness distribution. The fitness distribution has, in fact, a flat tail below the threshold f_c, which disappears only logarithmically in N as the system size is increased. In the limit $N \to \infty$ the distribution $\rho_i(f)$ is given by

$$\eta(f) = \frac{1}{1-f_c}\Theta(f-f_c), \quad (7)$$

with $f_c = 0.5371$. By substituting equation (7) in equation (6) we get $D_{asym}(N=\infty) = D_{asym} = 0.1543$ as estimation of the plateau in the thermodynamic limit. Turning back our attention to Figure 1, one can see that this result is consistent with the extrapolated numerical value $D_{asym} \sim 0.14(2)$.

We have also performed a similar analysis for the PBS model with a deterministic updating, in which the new fitnesses are obtained by iterating the logistic map, namely

$$f_i(t+1) = bf_i(t)(1-f_i(t)), \quad (8)$$

where $f_i(t)$ is the fitness of site i at time t, and b is a parameter set to the value 4 [17,18]. We observe the same qualitative behavior for all the quantities studied in the random updating case. The fitness distribution is however different since it is influenced by the invariant measure of the map, as pointed out in [17,18]. In the present case, the fitness distribution is strongly picked around $f=0$ and around $f=1$, for finite sizes N. The distribution is of course not symmetric and in the large N limit, all the f_j are above a threshold $f_c = 0.55(1)$ [23]. The value of the plateau D_{asym} converges, in the limit $N \to \infty$, to $D_{asym} \sim 0.15(2)$. By inserting the fitness distributions thus obtained in equation (6), we obtain an analytic estimation, $D_{asym} = 0.1556$, that is indeed very close to the random updating analytical value, and in agreement with our numerical results. This is reasonable, since the threshold of the parallel BS with deterministic rule is very near the threshold of the random updating case. Although we performed, for the computation of the plateau, simulations up to system size $N = 8000$, the need of a logarithmic extrapolation towards $N=\infty$ prevents us from obtaining a precise numerical estimation of the plateau. The values of the exponents α and μ show no substantial differences with respect to the random updating case, as is the case for the extremal BS model [19], thus confirming the robustness of α with respect to different updating rules. A more detailed analysis of both parallel and extremal BS models with different updating rules and different implementations of the initial perturbation will be reported elsewhere [14].

Summarizing, we have studied the features of damage spreading in the parallel version of the Bak-Sneppen model (PBS), with respect to its extremal version. In general, the damage spreading in the PBS model exhibits a faster dynamical evolution towards a stationary state. The estimation we get for the damage spreading exponent α in PBS is bigger that that of directed percolation. We propose that the origin of this discrepancy stems in the different choice of timescale for the two models. Indeed, after a suitable rescaling of the microscopical timescale in the PBS model, we get an exponent very close to the DP one.

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